On the Classical Radiation of Accelerated Electrons

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This paper is concerned with the properties of the radiation from a high energy accelerated electron, as recently observed in the General Electric synchrotron. An elementary derivation of the total rate of radiation is first presented, based on Larmor's formula for a slowly moving electron, and arguments of relativistic invariance. We then construct an expression for the instantaneous power radiated by an electron moving along an arbitrary, prescribed path. By casting this result into various forms, one obtains the angular distribution, the spectral distribution, or the combined angular and spectral distributions of the radiation. The method is based on an examination of the rate at which the electron irreversibly transfers energy to the electromagnetic field, as determined by half the difference of retarded and advanced electric field intensities. Formulas are obtained for an arbitrary charge-current distribution and then specialized to a point charge. The total radiated power and its angular distribution are obtained for an arbitrary trajectory. It is found that the direction of motion is a strongly preferred direction of emission at high energies. The spectral distribution of the radiation depends upon the detailed motion over a time interval large compared to the period of the radiation. However, the narrow cone of radiation generated by an energetic electron indicates that only a small part of the trajectory is effective in producing radiation observed in a given direction, which also implies that very high frequencies are emitted. Accordingly, we evaluate the spectral and angular distributions of the high frequency radiation by an energetic electron, in their dependence upon the parameters characterizing the instantaneous orbit. The average spectral distribution, as observed in the synchrotron measurements, is obtained by averaging the electron energy over an acceleration cycle. The entire spectrum emitted by an electron moving with constant speed in a circular path is also discussed. Finally, it is observed that quantum effects will modify the classical results here obtained only at extraordinarily large energies.

EARLY in 1945, much attention was focused on the design of accelerators for the production of very high energy electrons and other charged particles.[1] In connection with this activity, the author investigated in some detail the limitations to the attainment of high energy electrons imposed by the radiative energy loss[2] of the accelerated electrons. Although the results of this work were communicated to various interested persons,[1,3,4] no serious attempt at publication[5] was made. However, recent experiments on the radiation from the General Electric synchrotron[6] have made it desirable to publish the portion of the investigation that is concerned with the properties of the radiation from individual electrons, apart from the considerations on the practical attainment of very high energies. Accordingly, we derive various properties of the radiation from a high energy accelerated electron; the comparison with experiment has been given in the paper by Elder, Langmuir, and Pollock.

I. GENERAL FORMULAS

Before launching into the general discussion, it is well to notice an elementary derivation of the total rate of radiation, based on Larmor's classical formula for a slowly moving electron, and arguments of relativistic invariance. The Larmor formula for the power radiated by an electron that is instantaneously at rest is

$$P=\frac{2}{3}\frac{e^2}{c^3}\left(\frac{d\mathbf{v}}{dt}\right)^2=\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\mathbf{p}}{dt}\right)^2. \tag{I.1}$$

Now, radiated energy and elapsed time transform in the same manner under Lorentz transformations, whence the radiated power must be an invariant. We shall have succeeded in deriving a formula for the power radiated by an electron of arbitrary velocity if we can exhibit an invariant that reduces to (I.1) in the rest system of the electron. To accomplish this, we first replace the time derivative by the derivative with respect to the invariant proper time. The differential of proper time is defined by

$$ds^2=dt^2-1/c^2(dx^2+dy^2+dz^2),$$

or

$$ds=(1-v^2/c^2)^{\frac{1}{2}}dt. \tag{I.2}$$

Secondly, we replace the square of the proper time derivative of the momentum by the invariant combination

$$(d\mathbf{p}/ds)^2-1/c^2(dE/ds)^2.$$

Hence, as the desired invariant generalization of (I.1), we have

$$P=\frac{2}{3}\frac{e^2}{m^2c^3}\left[\left(\frac{d\mathbf{p}}{ds}\right)^2-\frac{1}{c^2}\left(\frac{dE}{ds}\right)^2\right]$$
$$=\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{E}{mc^2}\right)^2\left[\left(\frac{d\mathbf{p}}{dt}\right)^2-\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]. \tag{I.3}$$

The conventional form of this result is obtained on

[1] See L. I. Schiff, Rev. Sci. Inst. **17**, 6 (1946).
[2] D. Iwanenko and I. Pomeranchuk, Phys. Rev. **65**, 343 (1944).
[3] Edwin M. McMillan, Phys. Rev. **68**, 144 (1945).
[4] John P. Blewett, Phys. Rev. **69**, 87 (1946).
[5] Julian Schwinger, Phys. Rev. **70**, 798 (1946).
[6] Elder, Langmuir, and Pollock, Phys. Rev. **74**, 52 (1948).

writing

$$\mathbf{p}=\frac{m\mathbf{v}}{(1-\beta^2)^{\frac{1}{2}}},\quad E=\frac{mc^2}{(1-\beta^2)^{\frac{1}{2}}},\quad \beta=\frac{v}{c},$$

and performing the indicated differentiations. Thus

$$P=\frac{2}{3}\frac{e^2}{c^3}\frac{1}{(1-\beta^2)^3}\left[\left(\frac{d\mathbf{v}}{dt}\right)^2-\left(\frac{\mathbf{v}}{c}\times\frac{d\mathbf{v}}{dt}\right)^2\right]. \qquad (I.4)$$

The two important limiting cases of these formulas are realized in the linear accelerator and the synchrotron, or betatron. In the former, the rates of momentum and energy change are connected by

$$c^2p(dp/dt)=E(dE/dt),$$

whence

$$P=\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2=\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2, \qquad (I.5)$$

which shows that the radiated power depends only on the external force and is independent of the electron energy. The ratio of power lost in radiation to power gained from external sources is

$$P\Big/\frac{dE}{dt}=\frac{2}{3}\frac{e^2}{mc^2}\frac{dE/mc^2}{dx}, \qquad (I.6)$$

for high energy electrons. It is evident that radiative losses in a linear accelerator are negligible, unless the accelerating field supplies energy of the order mc^2 in a distance equal to the classical radius of the electron! For the circular trajectory of an electron in a synchrotron,

$$P=\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{E}{mc^2}\right)^2\left(\frac{d\mathbf{p}}{dt}\right)^2, \qquad (I.7)$$

since the energy changes slowly in comparison with the vectorial momentum. Now

$$\left(\frac{d\mathbf{p}}{dt}\right)^2=\omega_0{}^2p^2=\frac{\omega_0}{R}\beta^3E^2, \qquad (I.8)$$

where ω_0 and R are the instantaneous angular velocity and radius of curvature. Hence,

$$P=\tfrac{2}{3}\omega_0\frac{e^2}{R}\beta^3\left(\frac{E}{mc^2}\right)^4. \qquad (I.9)$$

For high energy electrons moving in a circular path, the energy radiated per revolution is

$$\delta E=(4\pi/3)(e^2/R)(E/mc^2)^4. \qquad (I.10)$$

A useful form of this result is

$$\delta E_{\mathrm{kev}}=88.5(E_{\mathrm{Bev}})^4/R_{\mathrm{met}} \qquad (I.11)$$

where E_{Bev} is the electron energy in units of 1 Bev $=10^9$ ev, R_{met} is the radius of the electron orbit in meters, and δE_{kev} is the energy radiated per revolution in units of 1 kev $=10^3$ ev.

We shall now construct an expression for the instantaneous power radiated by an electron moving along an arbitrary prescribed path. Our procedure will be such that by introducing various forms for this result the following additional physical quantities can be obtained: the angular distribution of the radiation, the rate of radiation into the various frequencies generated by the electron, and the angular distribution of the radiation emitted at each of these frequencies.

The method is based on a consideration of the rate at which the electron does work on the electromagnetic field,

$$-\int \mathbf{j}\cdot\mathbf{E}_{\mathrm{ret}}dv, \qquad (I.12)$$

which can be conveniently divided into two essentially different parts on writing

$$\mathbf{E}_{\mathrm{ret}}=\tfrac{1}{2}(\mathbf{E}_{\mathrm{ret}}+\mathbf{E}_{\mathrm{adv}})+\tfrac{1}{2}(\mathbf{E}_{\mathrm{ret}}-\mathbf{E}_{\mathrm{adv}}). \qquad (I.13)$$

Here $\mathbf{E}_{\mathrm{ret}}$ and $\mathbf{E}_{\mathrm{adv}}$ are the retarded and advanced electric field intensities generated by the electron charge and current densities, ρ and $\mathbf{j}$. The first part of (I.12), derived from the symmetrical combination of $\mathbf{E}_{\mathrm{ret}}$ and $\mathbf{E}_{\mathrm{adv}}$, changes sign on reversing the positive sense of time and therefore represents reactive power. It describes the rate at which the electron stores energy in the electromagnetic field, an inertial effect with which we are not concerned. However, the second part of (I.12), derived from the antisymmetrical combination of $\mathbf{E}_{\mathrm{ret}}$ and $\mathbf{E}_{\mathrm{adv}}$, remains unchanged on reversing the positive sense of time, and therefore represents resistive power. Subject to one qualification, it describes the rate of irreversible energy transfer to the electromagnetic field, which is the desired rate of radiation. Included in the second part of (I.12) are terms which have the form of the time derivative of an acceleration dependent electron energy. The latter is completely negligible compared with the electron kinetic energy for all realizable accelerations. It will be a simple manner to eliminate these unwanted terms after evaluating the dissipative part of (I.12). Thus, the power carried away by radiation is, provisionally,

$$P=-\int \mathbf{j}\cdot\mathbf{E}dv, \qquad (I.14)$$

with

$$\mathbf{E}=\tfrac{1}{2}(\mathbf{E}_{\mathrm{ret}}-\mathbf{E}_{\mathrm{adv}})=-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}-\nabla\phi. \qquad (I.15)$$

The expression of P in terms of the vector and scalar potentials, $\mathbf{A}$ and ϕ, can be simplified by employing the charge conservation equation

$$\nabla\cdot\mathbf{j}+\partial\rho/\partial t=0. \tag{I.16}$$

Thus,

$$P=\int\left[\frac{1}{c}\mathbf{j}\cdot\frac{\partial\mathbf{A}}{\partial t}-\rho\frac{\partial\phi}{\partial t}\right]dv+\frac{d}{dt}\int\rho\phi dv. \tag{I.17}$$

The second term of this formula is of the acceleration energy type and may be discarded. Hence the expression for the radiated power, which still includes unwanted acceleration energy terms, becomes

$$P=\int\left[\frac{1}{c}\mathbf{j}\cdot\frac{\partial\mathbf{A}}{\partial t}-\rho\frac{\partial\phi}{\partial t}\right]dv. \tag{I.18}$$

The retarded and advanced scalar potentials can be conveniently written as

$$\phi_{\mathrm{ret,\,adv}}(\mathbf{r},t)=\int\frac{\delta\left(t'-t\pm\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}',t')dv'dt', \tag{I.19}$$

where the upper sign is appropriate for the retarded potential. On introducing the Fourier integral representation of the function $\delta(t)$:

$$\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}d\omega, \tag{I.20}$$

the potentials assume the form

$$\phi_{\mathrm{ret,\,adv}}(\mathbf{r},t)=\frac{1}{2\pi}\int e^{i\omega(t'-t)}\times\frac{e^{\pm i(\omega/c)|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}',t')dv'dt'd\omega, \tag{I.21}$$

whence

$$\phi(\mathbf{r},t)=\frac{i}{2\pi}\int e^{i\omega(t'-t)}\times\frac{\sin\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}',t')dv'dt'd\omega. \tag{I.22}$$

Similarly,

$$\mathbf{A}_{\mathrm{ret,\,adv}}(\mathbf{r},t)=\int\frac{\delta\left(t'-t\pm\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}\times\frac{1}{c}\mathbf{j}(\mathbf{r}',t')dv'dt', \tag{I.23}$$

and

$$\mathbf{A}(\mathbf{r},t)=\frac{i}{2\pi}\int e^{i\omega(t'-t)}\frac{\sin(\omega/c)|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}\times\frac{1}{c}\mathbf{j}(\mathbf{r}',t')dv'dt'd\omega. \tag{I.24}$$

It will also prove useful to write

$$\frac{\sin(\omega/c)|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}=\frac{\omega}{c}\int\exp[i(\omega/c)\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')]\frac{d\Omega}{4\pi} \tag{I.25}$$

in which $d\Omega$ is an element of solid angle associated with the direction of the unit vector $\mathbf{n}$. The resultant expressions for ϕ and $\mathbf{A}$:

$$\phi(\mathbf{r},t)=\frac{i}{2\pi c}\int\exp[i(\omega/c)\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')]\times\rho(\mathbf{r}',t')dv'dt'\omega d\omega\frac{d\Omega}{4\pi},$$

$$\mathbf{A}(\mathbf{r},t)=\frac{i}{2\pi c}\int\exp[i(\omega/c)\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')]\times\frac{1}{c}\mathbf{j}(\mathbf{r}',t')dv'dt'\omega d\omega\frac{d\Omega}{4\pi} \tag{I.26}$$

are a superposition of plane waves traveling with the speed c.

The total radiated power, calculated from (18), (19), and (23), is

$$P(t)=\frac{1}{2}\int\left[\rho(\mathbf{r},t)\rho(\mathbf{r}',t')-\frac{1}{c^2}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r}',t')\right]\left[\frac{\delta'\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}-\frac{\delta'\left(t'-t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}\right]dvdv'dt', \tag{I.27}$$

where $\delta'(t)$ denotes the derivative of the delta-function. Alternatively, if the Fourier integral representations (I.22) and (I.24) are employed,

$$P(t)=-\frac{1}{2\pi}\int\Big[\rho(\mathbf{r},t)\rho(\mathbf{r}',t')-\frac{1}{c^2}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r}',t')\Big]e^{i\omega(t'-t)}\times\frac{\sin\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}dvdv'dt'\omega d\omega=\int_0^\infty P(\omega,t)d\omega. \tag{I.28}$$

It may be inferred that

$$P(\omega,t)=-\frac{\omega}{\pi}\int\Big[\rho(\mathbf{r},t)\rho(\mathbf{r}',t')-\frac{1}{c^2}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r}',t')\Big]\cos\omega(t-t')\cdot\frac{\sin\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}dvdv'dt' \tag{I.29}$$

represents the power radiated at the time t in a unit angular frequency range about ω. The latter statement is correct if $P(\omega, t)$ changes only slightly in a time interval equal to the reciprocal of ω, a condition that is adequately met in practice.

With the aid of (I.25), the total radiated power can also be written

$$P(t)=-\frac{1}{2\pi c}\int\Big[\rho(\mathbf{r},t)\rho(\mathbf{r}',t')-\frac{1}{c^2}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r}',t')\Big]\times\exp\Big[i\omega\Big[t'-t+\frac{1}{c}\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')\Big]\Big]\cdot\omega^2d\omega dvdv'dt'\frac{d\Omega}{4\omega}$$
$$=\frac{1}{c}\int\Big[\rho(\mathbf{r},t)\rho(\mathbf{r}',t')-\frac{1}{c^2}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r}',t')\Big]\times\delta''\Big(t'-t+\frac{1}{c}\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')\Big)\cdot dvdv'dt'\frac{d\Omega}{4\pi}=\int P(\mathbf{n},t)d\Omega. \tag{I.30}$$

Evidently,

$$P(\mathbf{n},t)=\frac{1}{4\pi c}\int\Big[\rho(\mathbf{r},\mathrm{t})\rho(\mathbf{r}',t')-\frac{1}{c^2}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r}',t')\Big]\times\delta''\Big(t'-t+\frac{1}{c}\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')\Big)dvdv'dt' \tag{I.31}$$

describes the power radiated per unit solid angle in the direction $\mathbf{n}$, at the time t. Similarly, (I.30) can be written

$$P(t)=\int_0^\infty d\omega\int d\Omega P(\mathbf{n},\omega,t), \tag{I.32}$$

where

$$P(\mathbf{n},\omega,t)=-\frac{1}{4\pi^2}\frac{\omega^2}{c}\int\Big[\rho(\mathbf{r},t)\rho(\mathbf{r}',t')-\frac{1}{c^2}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r}',t')\Big]\cdot\cos\omega\Big(t'-t+\frac{1}{c}\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')\Big)dvdv'dt' \tag{I.33}$$

represents the power radiated at time t into a unit solid angle about the direction $\mathbf{n}$ and contained in a unit angular frequency interval about the frequency ω. This interpretation is also subject to the adiabatic condition, that $P(\mathbf{n}, \omega, t)$ change only slightly in a time interval equal to the period of the radiation.

Thus far, we have dealt with the radiation of an arbitrary charge-current distribution. For a point electron of charge e, located at the variable position $\mathbf{R}(t)$,

$$\rho(\mathbf{r},t)=e\delta(\mathbf{r}-\mathbf{R}(t)),\quad \mathbf{j}(\mathbf{r},t)=e\mathbf{v}(t)\delta(\mathbf{r}-\mathbf{R}(t)), \tag{I.34}$$

where $\mathbf{v}(t)=d\mathbf{R}(t)/dt$. Our various formulas can now be simplified by performing the spatial integrations. Hence

$$P(t)=\frac{e^2}{2}\int_{-\infty}^{\infty}\Big(1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)\Big)\times\Big[\delta'\Big(\tau+\frac{|\mathbf{R}(t+\tau)-\mathbf{R}(t)|}{c}\Big)-\delta'\Big(\tau-\frac{|\mathbf{R}(t+\tau)-\mathbf{R}(t)|}{c}\Big)\Big]\times\frac{d\tau}{|\mathbf{R}(t+\tau)-\mathbf{R}(t)|}, \tag{I.35}$$

$$P(\mathbf{n}, t)=\frac{e^2}{4\pi c}\int_{-\infty}^{\infty}\left(1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)\right)\times\delta''\left(\tau-\frac{1}{c}\mathbf{n}\cdot(\mathbf{R}(t+\tau)-\mathbf{R}(t))\right)d\tau, \quad (I.36)$$

$$P(\omega, t)=-\frac{e^2\omega}{\pi}\int_{-\infty}^{\infty}\left(1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)\right)\times\cos\omega\tau\frac{\sin\frac{\omega}{c}|\mathbf{R}(t+\tau)-\mathbf{R}(t)|}{|\mathbf{R}(t+\tau)-\mathbf{R}(t)|}d\tau, \quad (I.37)$$

$$P(\mathbf{n}, \omega, t)=-\frac{e^2}{4\pi^2}\frac{\omega^2}{c}\int_{-\infty}^{\infty}\left(1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)\right)\times\cos\omega\left(\tau-\frac{1}{c}\mathbf{n}\cdot(\mathbf{R}(t+\tau)-\mathbf{R}(t))\right)d\tau, \quad (I.38)$$

in which we have placed $t'-t=\tau$.

The total power and its angular distribution can be obtained by straightforward integration, for an arbitrary electron trajectory. We consider, for example, the evaluation of $P(\mathbf{n}, t)$. It is convenient to introduce the variable

$$\gamma=\tau-\frac{1}{c}\mathbf{n}\cdot(\mathbf{R}(t+\tau)-\mathbf{R}(t))$$

for which

$$d\gamma=\left(1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}(t+\tau)\right)d\tau,$$

whence

$$P(\mathbf{n}, t)=\frac{e^2}{4\pi c}\int_{-\infty}^{\infty}\frac{1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)}{1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}(t+\tau)}\delta''(\gamma)d\gamma. \quad (I.39)$$

It is now merely necessary to integrate twice by parts and observe that $\gamma=0$ implies $\tau=0$, unless the electron velocity exceeds that of light (as in the Cerenkov effect). Thus

$$P(\mathbf{n}, t)=\frac{e^2}{4\pi c}\frac{1}{1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}(t)}\left[\frac{d}{d\tau}\frac{1}{1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}(t+\tau)}\times\frac{d}{d\tau}\frac{1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)}{1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}(t+\tau)}\right]_{\tau=0}. \quad (I.40)$$

On performing the indicated differentiations, we obtain

$$P(\mathbf{n}, t)=\frac{e^2}{4\pi c^3}\left[\frac{\dot{\mathbf{v}}^2}{\left(1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}\right)^3}+2\frac{\mathbf{n}\cdot\dot{\mathbf{v}}\mathbf{v}\cdot\dot{\mathbf{v}}}{\left(1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}\right)^4}-\frac{(\mathbf{n}\cdot\dot{\mathbf{v}})^2\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}\right)^5}\right]+\frac{d}{dt}\frac{e^2}{4\pi c^2}\left[-\frac{\frac{1}{c}\mathbf{v}\cdot\dot{\mathbf{v}}}{\left(1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}\right)^3}+\frac{\mathbf{n}\cdot\dot{\mathbf{v}}\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{1}{c}\mathbf{n}\cdot\mathbf{v}\right)}\right], \quad (I.41)$$

of which only the first term should be retained. Replacing the velocity by $\mathbf{v}=\mathbf{p}c^2/E$, we have finally,

$$P(\mathbf{n}, t)=\frac{e^2}{4\pi m^2c^3}\left(\frac{mc^2}{E}\right)^2\left[\frac{\dot{\mathbf{p}}^2-\frac{1}{c^2}\dot{E}^2}{\left(1-\mathbf{n}\cdot\frac{\mathbf{p}c}{E}\right)^3}-\left(\frac{mc^2}{E}\right)^2\frac{\left(\mathbf{n}\cdot\dot{\mathbf{p}}-\frac{1}{c}\dot{E}\right)^2}{\left(1-\mathbf{n}\cdot\frac{\mathbf{p}c}{E}\right)^5}\right]. \quad (I.42)$$

It can be verified that

$$P(t)=\int P(\mathbf{n}, t)d\Omega=\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{E}{mc^2}\right)^2\left[\dot{\mathbf{p}}^2-\frac{1}{c^2}\dot{E}^2\right],$$

in agreement with (I.3).

For the two extremes of linear acceleration and circular motion, the angular distribution of the radiated power is given by

$$P(\mathbf{n}, t)=\frac{e^2}{4\pi m^2c^3}\dot{p}^2(1-\beta^2)^3\frac{\sin^2\vartheta}{(1-\beta\cos\vartheta)^5}, \quad (I.43)$$

and

$$P(\mathbf{n},t)=\frac{e^2}{4\pi m^2c^3}\dot{\mathbf{p}}^2(1-\beta^2)\left[\frac{1}{(1-\beta\cos\vartheta)^3}-(1-\beta^2)\frac{\sin^2\vartheta\cos^2\varphi}{(1-\beta\cos\vartheta)^5}\right]. \quad (I.44)$$

The angles ϑ and φ are the polar angles of a coordinate system in which the z axis coincides with the instantaneous direction of motion, and the xz plane is that of the orbital motion. It is evident that at high speeds the direction of motion is a strongly preferred direction of emission. As a simple measure of this asymmetry we consider the mean value of $\sin^2\vartheta$, defined as

$$\langle\sin^2\vartheta\rangle=\int\sin^2\vartheta P(\mathbf{n},t)d\Omega/P(t). \quad (I.45)$$

An elementary calculation yields, for linear acceleration,

$$\langle\sin^2\vartheta\rangle=(1-\beta^2)\left[1-\tfrac{1}{2}(1-\beta^2)+\frac{3}{4}\frac{(1-\beta^2)^2}{\beta^5}\left(\log\frac{1+\beta}{1-\beta}-2\beta-\frac{2\beta^3}{3}\right)\right], \quad (I.46)$$

and, for circular motion,

$$\langle\sin^2\vartheta\rangle=(1-\beta^2)\left[1+\tfrac{1}{4}(1-\beta^2)-\frac{3}{4}\frac{1-\beta^2}{\beta^3}\left(\log\frac{1+\beta}{1-\beta}-2\beta\right)-\frac{3}{8}\frac{(1-\beta^2)^2}{\beta^5}\left(\log\frac{1+\beta}{1-\beta}-2\beta-\frac{2\beta^3}{3}\right)\right]. \quad (I.47)$$

The values of these averages, for low energy electrons, are 4/5 and 3/5, for linear and circular motion, respectively. Both types of motion yield the following limiting form for high energy electrons

$$\langle\sin^2\vartheta\rangle=1-\beta^2=(mc^2/E)^2, \quad (I.48)$$

and therefore the mean angle between the direction of emission and that of the electron's motion is

$$\bar{\vartheta}=\langle\vartheta^2\rangle^{\frac{1}{2}}=mc^2/E,\quad E/mc^2\gg 1. \quad (I.49)$$

A useful form of this result is

$$\bar{\vartheta}_{\min}=1.76/E_{\mathrm{Bev}},$$

where $\bar{\vartheta}_{\min}$ is the mean angle of emission in minutes and E_{Bev} is the electron energy in units of Bev.

The power radiated into a given frequency range necessarily depends upon the detailed motion of the electron over a time interval that is large compared with the period of radiation under consideration. However, the very narrow cone of radiation that is generated by a high energy electron suggests that only a small part of the electron trajectory is effective in producing the radiation observed in a given direction, which also implies that very high frequencies must be emitted. We therefore consider first the high frequency radiation generated by an energetic electron, in the anticipation that only the instantaneous nature of the electron's motion will be involved. Secondly, we shall derive the entire spectrum emitted by an electron moving with constant speed in a circular path.

II. HIGH FREQUENCY RADIATION BY ENERGETIC ELECTRONS

To evaluate (I.37) for $E/mc^2\gg 1$, it is sufficient to write

$$\begin{aligned}|\mathbf{R}(t+\tau)-\mathbf{R}(t)|&=|\tau\mathbf{v}(t)+(\tau^2/2)\dot{\mathbf{v}}(t)+(\tau^3/6)\partial^2\mathbf{v}(t)/\partial t^2|\\&=(\tau^2v^2+\tau^3\mathbf{v}\cdot\dot{\mathbf{v}}+(\tau^4/4)\dot{\mathbf{v}}^2+(\tau^4/3)\mathbf{v}\cdot\partial^2\mathbf{v}/\partial t^2)^{\frac{1}{2}}.\end{aligned} \quad (II.1)$$

Furthermore, $\mathbf{v}\cdot\dot{\mathbf{v}}=(d/dt)\frac{1}{2}\mathbf{v}^2$ may be placed equal to zero since v differs negligibly from c when $E/mc^2\gg 1$. In a similar manner, $\mathbf{v}\cdot\partial^2\mathbf{v}/\partial t^2\simeq-\dot{\mathbf{v}}^2\simeq-c^4/R^2$. In the latter formula R is the instantaneous radius of curvature. Hence

$$|\mathbf{R}(t+\tau)-\mathbf{R}(t)|\simeq v|\tau|-\frac{1}{24}\frac{c^3}{R^2}|\tau^3| \quad (II.2)$$

and

$$P(\omega,t)=-\frac{2}{\pi}\frac{e^2\omega}{c}\int_0^\infty\left(1-\beta^2+\frac{1}{2}\frac{c^2\tau^2}{R^2}\right)\times\cos\omega\tau\sin\omega\left(\beta\tau-\frac{c^2\tau^3}{24R^2}\right)\frac{d\tau}{\tau}. \quad (II.3)$$

In writing this result, an approximation similar to (II.2) has been employed on the velocity dependent factor of (I.37):

$$1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)=1-\frac{v^2}{c^2}-\frac{\tau}{c^2}\mathbf{v}\cdot\dot{\mathbf{v}}-\frac{\tau^2}{2c^2}\mathbf{v}\cdot\partial^2\mathbf{v}/\partial t^2\simeq 1-\beta^2+\frac{1}{2}\frac{c^2\tau^2}{R^2}. \quad (II.4)$$

In addition, the first term of (II.2) suffices for the evaluation of the denominator in (I.37). It is now convenient to write, approximately,

$$P(\omega,t)=\frac{1}{\pi}\frac{e^2\omega}{c}\int_0^\infty\left(1-\beta^2+\frac{1}{2}\frac{c^2\tau^2}{R^2}\right)\times\left\{\sin\left(\omega(1-\beta)\tau+\frac{\omega c^2}{24R^2}\tau^3\right)\frac{d\tau}{\tau}-\sin 2\omega\tau\frac{d\tau}{\tau}\right\}. \quad \text{(II.5)}$$

The approximation employed in the second term of (II.5) is valid provided $\omega \gg c/R$, that is, if we restrict attention to frequencies that greatly exceed the instantaneous angular frequency of the electron. To justify this statement, note that the largest values of τ that contribute to the integral in question are $\tau\sim 1/\omega$. Thus the neglected term is of the order of magnitude $\omega\tau^3c^2/R^2\sim(c/\omega R)^2\ll 1$.

To simplify the first integral in (II.5), we write

$$\tau=\frac{2R}{c}(1-\beta^2)^{\frac{1}{2}}x, \quad \text{(II.6)}$$

whence

$$P(\omega,t)=\frac{1}{\pi}\frac{e^2\omega}{c}(1-\beta^2)\left[\int_0^\infty(1+2x^2)\times\sin\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)\frac{dx}{x}-\frac{\pi}{2}\right]. \quad \text{(II.7)}$$

In this formula, we have placed

$$\xi=\frac{2}{3}\frac{\omega R}{c}(1-\beta^2)^{\frac{3}{2}}=\frac{2}{3}\frac{\omega R}{c}\left(\frac{mc^2}{E}\right)^3 \quad \text{(II.8)}$$

and employed the well-known integral

$$\int_0^\infty\frac{\sin ax}{x}dx=\frac{\pi}{2},\quad a>0. \quad \text{(II.9)}$$

The approximation (II.2) can now be justified by considering the range of variable that is of most importance for the integral in (II.7). The essential frequencies are such that $\xi\sim 1$, or $\omega\sim(c/R)(E/mc^2)^3$. Clearly, then, the important values of x are $x\sim 1$, or $\tau\sim(R/c)(mc^2/E)$. The series (II.7) is essentially an expansion in powers of $(c\tau/R)^2\sim(mc^2/E)^2\ll 1$. Thus the neglect of higher terms in the expansion is justified provided $E/mc^2\gg 1$.

The integral contained in (II.7) can be recognized as related to the Airy integral[7]

$$\int_0^\infty\cos\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)dx=3^{-\frac{1}{2}}K_{1/3}(\xi). \quad \text{(II.10)}$$

Now

$$3^{-\frac{1}{2}}K_{1/3}(\xi)=\frac{d}{d\xi}\int_0^\infty\sin\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)\frac{dx}{x},$$

whence

$$3^{-\frac{1}{2}}\int_\xi^\infty K_{1/3}(\eta)d\eta=\lim_{L\to\infty}\int_0^\infty\sin L(x+\tfrac{1}{3}x^3)\frac{dx}{x}-\int_0^\infty\sin\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)\frac{dx}{x}=\frac{\pi}{2}-\int_0^\infty\sin\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)\frac{dx}{x}. \quad \text{(II.11)}$$

Furthermore,

$$3^{-\frac{1}{2}}K_{2/3}(\xi)=\int_0^\infty x\sin\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)dx \quad \text{(II.12)}$$

since

$$\left(\frac{d}{d\xi}+\frac{1}{3\xi}\right)K_{1/3}(\xi)=-K_{2/3}(\xi). \quad \text{(II.13)}$$

Hence

$$\int_0^\infty(1+2x^2)\sin\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)\frac{dx}{x}-\frac{\pi}{2}=3^{-\frac{1}{2}}\left(2K_{2/3}(\xi)-\int_\xi^\infty K_{1/3}(\eta)d\eta\right)=3^{-\frac{1}{2}}\int_\xi^\infty K_{5/3}(\eta)d\eta. \quad \text{(II.14)}$$

The last simplification involves the recurrence relation

$$2\frac{d}{d\xi}K_{2/3}(\xi)+K_{1/3}(\xi)=-K_{5/3}(\xi). \quad \text{(II.15)}$$

Our final result is

$$P(\omega,t)=\frac{3^{\frac{1}{2}}}{4\pi}\frac{e^2}{R}\left(\frac{E}{mc^2}\right)^4\frac{\omega_0\omega}{\omega_c^2}\int_{\omega/\omega_c}^\infty K_{5/3}(\eta)d\eta, \quad \text{(II.16)}$$

where the critical frequency ω_c is defined by

$$\omega_c=\tfrac{3}{2}\omega_0\left(\frac{E}{mc^2}\right)^3 \quad \text{(II.17)}$$

and

$$\omega_0=c/R.$$

Before proceeding with the discussion of the formula, it is advisable to check our approximations by verifying that

$$P(t)=\int_0^\infty P(\omega,t)d\omega=\tfrac{2}{3}\omega_0\frac{e^2}{R}\left(\frac{E}{mc^2}\right)^4,$$

[7] G. N. Watson, *Bessel Functions* (The Macmillan Company, New York, 1945), p. 188.

or that

$$\int_0^\infty \xi d\xi \int_\xi^\infty K_{5/3}(\eta)d\eta = \frac{1}{2}\int_0^\infty \xi^2 K_{5/3}(\xi)d\xi = \frac{8\pi}{3^{7/3}}.$$

This is indeed the value yielded by the formula

$$\int_0^\infty \xi^{\mu-1}K_\nu(\xi)d\xi = 2^{\mu-2}\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right). \quad \text{(II.18)}$$

In virtue of the properties of the Bessel functions of imaginary argument, the behavior of $P(\omega, t)$ is radically different, depending upon whether ω/ω_c is large or small compared with unity. For $\omega \ll \omega_c$, it is convenient to write

$$P(\omega, t) = \frac{3^{\frac{1}{2}}}{4\pi}\frac{e^2}{R}\left(\frac{E}{mc^2}\right)^4 \frac{\omega_0\omega}{\omega_c^2} \times\left[2K_{2/3}\left(\frac{\omega}{\omega_c}\right) + \int_0^{\omega/\omega_c} K_{1/3}(\eta)d\eta - \frac{\pi}{3^{\frac{1}{2}}}\right] \quad \text{(II.19)}$$

and insert the power series expansions for small values of the argument. Thus,

$$\omega/\omega_c \ll 1 : P(\omega, t) = \frac{3^{1/6}\Gamma(\frac{2}{3})}{\pi}\frac{e^2}{R}\left(\frac{\omega}{\omega_0}\right)^{\frac{1}{3}} \times\left[1 - \frac{\Gamma(\frac{2}{3})}{2}\left(\frac{\omega}{2\omega_c}\right)^{\frac{4}{3}} + \cdots\right], \quad \text{(II.20)}$$

of which the leading term for small frequencies varies as $\omega^{\frac{1}{3}}$ and is independent of the electron energy. If $\omega/\omega_c \gg 1$, we may employ the asymptotic form for large argument in Eq. (II.16), whence

$$\omega/\omega_c \gg 1 : P(\omega, t) = \frac{3}{4}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}\frac{e^2}{R} \times\left(\frac{E}{mc^2}\right)^4 \frac{\omega_0}{\omega_c}\left(\frac{\omega}{\omega_c}\right)^{\frac{1}{2}} e^{-\omega/\omega_c} \times\left[1 + \frac{55}{72}\frac{\omega_c}{\omega} + \cdots\right]. \quad \text{(II.21)}$$

Therefore, the energy radiated into a unit frequency interval steadily increases with frequency until $\omega/\omega_c \sim 1$, after which there is a rapid decrease.[8] Since the power radiated at frequencies not in excess of ω_c is independent of E, but varies as $\omega^{\frac{1}{3}}$, the total power should vary as $\omega_c^{4/3}$, which is indeed proportional to $(E/mc^2)^4$.

Associated with the critical frequency is a wave-length

$$\lambda_c = (4\pi/3)R(mc^2/E)^3 \quad \text{(II.22)}$$

which may be written, for practical purposes, as

$$\lambda_{c,\,A} = 5.59(R_{\text{met}}/(E_{\text{Bev}})^3). \quad \text{(II.23)}$$

Here R_{met} and E_{Bev} have the meanings previously attributed to them, while $\lambda_{c,\,A}$ denotes the critical wave-length in units of $1A = 10^{-8}$ cm. Evidently an electron accelerator will be a source of high frequency radiation. It is desirable to have a simply physical explanation for the very high frequencies generated by an energetic electron. We shall base such an explanation upon the angular distribution properties of the radiation. Since the mean angle between the direction of motion and that of the radiation is $\bar{\vartheta} = mc^2/E$, the time interval during which radiation is emitted toward the observer is

$$\Delta t \sim (R/c)\bar{\vartheta}$$

being the time required for the direction of motion to move through the angle $\bar{\vartheta}$. However, the time interval for reception of the pulse by the observer differs from Δt, in virtue of the Doppler effect. The time of reception, at the point $\mathbf{r}$, of a signal generated at the time t and the point $\mathbf{R}(t)$ is

$$t' = t + \frac{|\mathbf{r} - \mathbf{R}(t)|}{c},$$

whence

$$dt' = \left(1 - \frac{\mathbf{r} - \mathbf{R}(t)}{|\mathbf{r} - \mathbf{R}(t)|}\cdot\frac{\mathbf{v}(t)}{c}\right)dt = (1 - \beta\cos\vartheta)dt.$$

Thus the time interval during which the pulse is received is of the order of magnitude

$$\Delta t' \sim (1 - \beta + \tfrac{1}{2}\bar{\vartheta}^2)\Delta t \sim (1 - \beta^2)\Delta t \sim (R/c)(mc^2/E)^3$$

and the Fourier spectrum of the pulse will contain all frequencies up to a maximum of the order of magnitude

$$\omega_c \sim 1/\Delta t' \sim (c/R)(E/mc^2)^3,$$

in agreement with our more precise considerations.

The combined spectral and angular distribution for a high energy electron can be obtained in a similar way. We write, approximately,

$$P(\mathbf{n}, \omega, t) = -\frac{e^2}{4\pi^2}\frac{\omega^2}{c}\int_{-\infty}^{\infty} \times\left(1 - \beta^2 + \frac{1}{2}\frac{c^2\tau^2}{R^2}\right)\cos\omega\left(\left(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}\right)\tau - \frac{1}{2}\frac{\mathbf{n}\cdot\dot{\mathbf{v}}}{c}\tau^2 - \tfrac{1}{6}\mathbf{n}\cdot\frac{\partial^2\mathbf{v}/\partial t^2}{c}\tau^3\right)d\tau. \quad \text{(II.24)}$$

It will be convenient to replace the polar angles ϑ and φ by the angles ψ and χ. The former designates

[8] These qualitative results have also been obtained by L. Arzimovich and I. Pomeranchuk, J. Phys. USSR 9, 267 (1945).

the angle between the direction of emission and the instantaneous orbital plane, while χ denotes the angle, projected onto the orbital plane, between the direction of emission and direction of motion. The connection between the two sets of spherical coordinates is

$$\begin{aligned}\sin\vartheta\,\sin\varphi&=\sin\psi,\\ \sin\vartheta\,\cos\varphi&=\cos\psi\,\sin\chi,\\ \cos\vartheta&=\cos\psi\,\cos\chi.\end{aligned}\tag{II.25}$$

Hence,

$$\begin{aligned}\frac{\mathbf{n}\cdot\mathbf{v}}{c}&=\frac{c}{R}\sin\vartheta\,\cos\varphi\simeq\frac{c}{R}\chi,\\ 1-\frac{\mathbf{n}\cdot\mathbf{v}}{c}&=1-\beta\cos\vartheta\simeq\tfrac{1}{2}(1-\beta^2+\psi^2+\chi^2),\\ \frac{\mathbf{n}\cdot(\partial^2\mathbf{v}/\partial t^2)}{c}&\simeq\frac{1}{c^2}\mathbf{v}\cdot(\partial^2\mathbf{v}/\partial t^2)\simeq-\frac{c^2}{R^2},\end{aligned}\tag{II.26}$$

in which approximations have been introduced based on the small angle between the directions of emission and motion. Expressed in these angular coordinates, (II.24) becomes

$$P(\mathbf{n},\omega,t)=-\frac{e^2}{4\pi^2}\frac{\omega^2}{c}\int_{-\infty}^{\infty}\times\left(1-\beta^2+\frac{1}{2}\frac{c^2\tau^2}{R^2}\right)\cos\omega\times\left(\tfrac{1}{2}(1-\beta^2+\psi^2+\chi^2)\tau-\frac{c\chi}{2R}\tau^2+\frac{c^2\tau^3}{6R^2}\right)d\tau.\tag{II.27}$$

To bring this integral into the standard Airy integral form, it is necessary to eliminate the term in τ^2 that occurs in the argument of the trigonometric function. We achieve this by the following substitution for τ:

$$\tau\rightarrow\tau+\frac{R}{c}\chi,\tag{II.28}$$

whence

$$P(\mathbf{n},\omega,t)=-\frac{e^2}{4\pi^2}\frac{\omega^2}{c}\int_{-\infty}^{\infty}\times\left(1-\beta^2+\tfrac{1}{2}\chi^2+\frac{1}{2}\frac{c^2\tau^2}{R^2}+\frac{c}{R}\chi\tau\right)\cos\omega\left(\tfrac{1}{2}(1-\beta^2+\psi^2)\tau+\frac{c^2}{6R^2}\tau^3+\frac{R}{2c}(1-\beta^2+\psi^2)\chi+\frac{R}{6c}\chi^3\right)d\tau.\tag{II.29}$$

It should now be remarked that only the distribution in the angle ψ is of interest, that in the angle χ being essentially unobservable in practice. Accordingly, we may integrate (II.29) with respect to χ to obtain $P(\psi,\omega,t)$, the power emitted per unit angle relative to the orbital plane, and per unit angular frequency, at the time t. The introduction of the variables

$$x=(1-\beta^2+\psi^2)^{-\frac{1}{2}}\chi,\quad y=(1-\beta^2+\psi^2)^{-\frac{1}{2}}\frac{c\tau}{R}\tag{II.30}$$

yields

$$P(\psi,\omega,t)=-\frac{1}{4\pi^2}\frac{e^2}{R}\frac{\omega^2}{\omega_0^2}(1-\beta^2+\psi^2)\int_{-\infty}^{\infty}\times\left(1-\beta^2+\frac{1-\beta^2+\psi^2}{2}(x^2+y^2+2xy)\right)\cdot\cos\left[\frac{\omega R}{2c}(1-\beta^2+\psi^2)^{\frac{3}{2}}(x+\tfrac{1}{3}x^3+y+\tfrac{1}{3}y^3)\right]dxdy,\tag{II.31}$$

whence

$$P(\psi,\omega,t)=\frac{1}{3\pi^2}\frac{e^2}{R}\frac{\omega^2}{\omega_0^2}(1-\beta^2+\psi^2)^2\times\left[K_{2/3}{}^2\left(\frac{\omega R}{3c}(1-\beta^2+\psi^2)^{\frac{3}{2}}\right)+\frac{\psi^2}{1-\beta^2+\psi^2}K_{1/3}{}^2\left(\frac{\omega R}{3c}(1-\beta^2+\psi^2)^{\frac{3}{2}}\right)\right]\tag{II.32}$$

on employing (II.10), (II.12), and the identity

$$\int_0^{\infty}(1+x^2)\cos\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)dx=\frac{2}{3\xi}\int_0^{\infty}d\sin\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)=0.\tag{II.33}$$

This result can also be written

$$P(\psi,\omega,t)=\frac{3}{4\pi^2}\frac{e^2}{R}\left(\frac{\omega}{\omega_c}\right)^2\times\left(\frac{E}{mc^2}\right)^2\left(1+\left(\frac{E}{mc^2}\psi\right)^2\right)^2\cdot\left[K_{2/3}{}^2(\xi)+\frac{\left(\frac{E}{mc^2}\psi\right)^2}{1+\left(\frac{E}{mc^2}\psi\right)^2}K_{1/3}{}^2(\xi)\right],\tag{II.34}$$

where

$$\xi=\frac{\omega}{2\omega_c}\left(1+\left(\frac{E}{mc^2}\psi\right)^2\right)^{\frac{3}{2}}. \quad \text{(II.35)}$$

Evidently the spectrum of the radiation emitted at the angle ψ extends up to a maximum frequency

$$\sim\omega_c\left(1+\left(\frac{E}{mc^2}\psi\right)^2\right)^{-\frac{3}{2}},$$

which decreases rapidly with increasing angle if $\psi>mc^2/E$. This again implies that appreciable radiation occurs only at angles $\psi\sim mc^2/E$, which is made evident by the angular distribution function for the radiation, irrespective of frequency:

$$P(\psi,t)=\int_0^\infty P(\psi,\omega,t)d\omega$$

$$=\omega_0\frac{e^2}{R}\left(\frac{E}{mc^2}\right)^5\left(1+\left(\frac{E}{mc^2}\psi\right)^2\right)^{-5/2}$$

$$\times\left[\frac{7}{16}+\frac{5}{16}\frac{\left(\frac{E}{mc^2}\psi\right)^2}{1+\left(\frac{E}{mc^2}\psi\right)^2}\right], \quad \text{(II.36)}$$

obtained from (II.34) with the aid of the integral

$$\int_0^\infty K_\mu^2(x)x^2dx=\frac{\pi^2}{4}\frac{\frac{1}{4}-\mu^2}{\cos\pi\mu}. \quad \text{(II.37)}$$

This result agrees, of course, with that computed from $P(\mathbf{n},t)$ by integrating with respect to χ and introducing the approximations appropriate to the situation under discussion.

The synchrotron spectral measurements reported in reference 6 pertain, not to the instantaneous power spectrum individually emitted by the accelerated electrons, but rather to the spectrum averaged over an acceleration cycle. For the high energies under consideration, the electron energy varies linearly with the magnetic field. If the latter increases sinusoidally to its maximum value in an interval T,

$$E(t)=E_m\sin\frac{\pi}{2}\frac{t}{T}, \quad \text{(II.38)}$$

where E_m is the maximum electron energy. Correspondingly,

$$\omega_c(t)=\omega_m\sin^3\frac{\pi}{2}\frac{t}{T}, \quad \text{(II.39)}$$

with

$$\omega_m=(3/2)\omega_0(E_m/mc^2)^3. \quad \text{(II.40)}$$

The desired average power spectrum is given by

$$\bar{P}(\omega)=\frac{1}{T}\int_0^T P(\omega,t)dt$$

$$=\frac{3^{\frac{1}{2}}}{4\pi}\frac{e^2}{R}\frac{1}{T}\int_0^T dt\left(\frac{E(t)}{mc^2}\right)^4$$

$$\times\frac{\omega_0\omega}{\omega_c^2(t)}\int_{\omega/\omega_c(t)}^\infty K_{5/3}(\eta)d\eta. \quad \text{(II.41)}$$

It is convenient to introduce as a new variable

$$\tau=\omega_m/\omega_c(t)=\sin^{-3}\frac{\pi}{2}\frac{t}{T}. \quad \text{(II.42)}$$

With the aid of the differential relation

$$\frac{dt}{T}=\frac{2}{3\pi}(\tau^{\frac{2}{3}}-1)^{-\frac{1}{2}}\frac{d\tau}{\tau},$$

(II.41) now reads

$$\bar{P}(\omega)=\frac{3^{\frac{1}{2}}}{2\pi^2}\frac{e^2}{R}\left(\frac{E_m}{mc^2}\right)^4\frac{\omega_0\omega}{\omega_m^2}$$

$$\times\int_1^\infty\frac{\tau^{-\frac{1}{3}}}{(\tau^{\frac{2}{3}}-1)^{\frac{1}{2}}}d\tau\int_{(\omega/\omega_m)\tau}^\infty K_{5/3}(\eta)d\eta. \quad \text{(II.43)}$$

The integrals can be simplified with an integration by parts, according to

$$3\int_1^\infty d(\tau^{\frac{2}{3}}-1)^{\frac{1}{2}}\int_{(\omega/\omega_m)\tau}^\infty K_{5/3}(\eta)d\eta$$

$$=3\frac{\omega}{\omega_m}\int_1^\infty(\tau^{\frac{2}{3}}-1)^{\frac{1}{2}}K_{5/3}\left(\frac{\omega}{\omega_m}\tau\right)d\tau.$$

Therefore

$$\bar{P}(\omega)=\frac{3^{\frac{3}{2}}}{2\pi^2}\frac{e^2}{R}\left(\frac{E_m}{mc^2}\right)^4\frac{\omega_0}{\omega_m}F\left(\frac{\omega}{\omega_m}\right) \quad \text{(II.44)}$$

or

$$\bar{P}(\omega)=\frac{3^{\frac{1}{2}}}{\pi^2}\frac{e^2}{R}\frac{E_m}{mc^2}F\left(\frac{\omega}{\omega_m}\right), \quad \text{(II.45)}$$

where

$$F(x)=x^2\int_1^\infty(\tau^{\frac{2}{3}}-1)^{\frac{1}{2}}K_{5/3}(x\tau)d\tau. \quad \text{(II.46)}$$

It can be verified that the total time average power, computed from (II.44) as $\int_0^\infty\bar{P}(\omega)d\omega$, agrees with the more direct calculation:

$$\bar{P}=\tfrac{2}{3}\omega_0\frac{e^2}{R}\left(\frac{E_m}{mc^2}\right)^4\frac{1}{T}\int_0^T \sin^4\frac{\pi}{2}\frac{t}{T}dt$$
$$=\tfrac{1}{4}\omega_0\frac{e^2}{R}\left(\frac{E_m}{mc^2}\right)^4. \tag{II.47}$$

The asymptotic form of $F(x)$ for $x\gg1$ can be computed in a straightforward manner from the known behavior of $K_{5/3}$ for large argument, with the result:

$$x\gg1: F(x)\sim\frac{\pi}{2}3^{-\frac{1}{2}}e^{-x}\left(1+\frac{7}{18}\frac{1}{x}+\cdots\right). \tag{II.48}$$

The first term in the expansion of $F(x)$ for small x is obtained from the leading term in the expansion of $K_{5/3}$:

$$x\ll1: F(x)=2^{\frac{2}{3}}\Gamma(5/3)x^{\frac{1}{3}}$$
$$\times\int_1^\infty(\tau^{\frac{2}{3}}-1)^{\frac{1}{2}}\tau^{-5/3}d\tau=2^{-\frac{1}{3}}\pi\Gamma(\tfrac{2}{3})x^{\frac{1}{3}}. \tag{II.49}$$

The corresponding average power spectrum,

$$\omega\ll\omega_m: \bar{P}(\omega)=\frac{3^{1/6}}{\pi}\Gamma(\tfrac{2}{3})\frac{e^2}{R}\left(\frac{\omega}{\omega_0}\right)^{\frac{1}{3}},$$

agrees with the first term of (II.20), the energy independent low frequency instantaneous power spectrum. To obtain further terms in an expansion, we observe that

$$\frac{d}{dx}(x^{-\frac{1}{3}}F(x))=-x^{5/3}\int^\infty\tau(\tau^{\frac{2}{3}}-1)^{\frac{1}{2}}K_{2/3}(x\tau)d\tau \tag{II.50}$$

since

$$\frac{d}{dx}(x^{5/3}K_{5/3}(x))=-x^{5/3}K_{2/3}(x). \tag{II.51}$$

By an appropriate change in variable, this becomes

$$\frac{d}{dx}(x^{-\frac{1}{3}}F(x))=-x^{-\frac{2}{3}}\int_x^\infty\xi(\xi^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{1}{2}}K_{2/3}(\xi)d\xi, \tag{II.52}$$

and an integration with respect to x, in conjunction with (II.49), yields

$$x^{-\frac{1}{3}}F(x)-2^{-\frac{1}{3}}\pi\Gamma(\tfrac{2}{3})$$
$$=-\int_0^x\eta^{-\frac{2}{3}}d\eta\int_\eta^\infty\xi(\xi^{\frac{2}{3}}-\eta^{\frac{2}{3}})^{\frac{1}{2}}K_{2/3}(\xi)d\xi. \tag{II.53}$$

For small x, the leading terms on the right side of (II.53) are

$$-3x^{\frac{1}{3}}\int_0^\infty\xi^{4/3}K_{2/3}(\xi)d\xi+\tfrac{1}{2}x\int_0^\infty\xi^{\frac{2}{3}}K_{2/3}(\xi)d\xi$$
$$=-\frac{3}{2^{\frac{2}{3}}}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{5}{6})x^{\frac{1}{3}}+2^{-4/3}\Gamma(\tfrac{1}{2})\Gamma(7/6)x,$$

whence

$$x\ll1: F(x)=2^{-\frac{1}{3}}\pi\Gamma(\tfrac{2}{3})x^{\frac{1}{3}}$$
$$\times\left[1-\frac{3}{2^{\frac{1}{3}}\pi^{\frac{1}{2}}}\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3})}x^{\frac{1}{3}}+\frac{1}{2\pi^{\frac{1}{2}}}\frac{\Gamma(7/6)}{\Gamma(\frac{2}{3})}x+\cdots\right]. \tag{II.54}$$

The approximate evaluations of $F(x)$ for large and small argument furnish a reasonable qualitative picture of the function. To obtain the complete curve, resort must be had to numerical integration. The result may be seen in Fig. 1 of reference 6, which is essentially a plot of $F(x)$.

We shall conclude this section by briefly examining under what conditions quantum phenomena will invalidate the classical considerations we have presented. This will occur when the momentum of the emitted quantum is comparable with the electron momentum. Hence, for the validity of our classical treatment, it is required that

$$\hbar\omega\ll E,\quad \omega\sim\omega_c,$$

or

$$E/mc^2\ll\left(\frac{R}{\hbar/mc}\right)^{\frac{1}{2}}. \tag{II.55}$$

For a given radius, say $R=10^2$ cm, this sets an upper limit to the electron energy, of the order of 10^{12} ev. However, it must be remembered that in a magnetic device, the radius of the orbit is related to the maximum particle energy, as restricted by the strength of attainable magnetic fields:

$$E=eHR.$$

The limitation (II.55) should thus be written

$$\frac{E}{mc^2}\ll\frac{mc^2}{(e\hbar/mc)H}, \tag{II.56}$$

which implies that, for $H=10^4$ gauss, classical theory will be adequate if the particle energy does not exceed 10^{15} ev.

III. RADIATION BY AN ELECTRON IN UNIFORM CIRCULAR MOTION

In order to obtain the complete spectrum radiated by an accelerated electron, it is necessary to specify the entire trajectory. Accordingly, we shall apply our general method to calculate the radiation properties of an electron moving with constant speed in a circular path. Since the motion is periodic, the spectrum will consist of harmonics of the rotational angular frequency $\omega_0 = v/R$. We shall obtain the total power radiated into each harmonic as well as its angular distribution.[9]

The quantities entering into the formula (I.37) for the spectral distribution are easily evaluated for circular motion. Thus

$$\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)=v^2\cos\omega_0\tau, \tag{III.1}$$

and

$$|\mathbf{R}(t+\tau)-\mathbf{R}(t)|=2R\left|\sin\frac{\omega_0\tau}{2}\right|\cdot \tag{III.2}$$

Furthermore, the periodic nature of the motion can be exploited by writing

$$\omega_0\tau=\varphi+2\pi k, \tag{III.3}$$

and replacing the integration with respect to τ from $-\infty$ to $+\infty$ by one with respect to φ from $-\pi$ to $+\pi$, combined with a summation with respect to the integer k from $-\infty$ to ∞. Thus,

$$P(\omega)=-\frac{e^2}{\pi}\frac{\omega}{\omega_0}\sum_{k=-\infty}^{\infty}\cos 2\pi\frac{\omega}{\omega_0}k \times\int_{-\pi}^{\pi}(1-\beta^2\cos\varphi)\cos\frac{\omega}{\omega_0}\varphi \times\frac{\sin[(2\omega R/c)\sin(\varphi/2)]}{2R\sin\varphi/2}d\varphi. \tag{III.4}$$

However, according to the Poisson sum formula,

$$\sum_{k=-\infty}^{\infty}\cos 2\pi\frac{\omega}{\omega_0}k=\sum_{n=-\infty}^{\infty}\delta\left(\frac{\omega}{\omega_0}-n\right), \tag{III.5}$$

in which n ranges over all integral values. Since ω is restricted to be positive, we obtain

$$P(\omega)=\sum_{n=1}^{\infty}\delta(\omega-n\omega_0)P_n \tag{III.6}$$

with

$$P_n=-n\omega_0\frac{e^2}{R}\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\beta^2\cos\varphi)\cos n\varphi \times\frac{\sin(2n\beta\sin\varphi/2)}{\sin\varphi/2}d\varphi. \tag{III.7}$$

The discrete nature of the spectrum is now exhibited, with P_n representing the power radiated into the nth harmonic. The latter can be rewritten as

$$P_n=-n\omega_0\frac{e^2}{R}\left[(1-\beta^2)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(2n\beta\sin\varphi/2)}{\sin\varphi/2}\cos n\varphi d\varphi +2\beta^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(2n\beta\sin\varphi/2)\times\sin\varphi/2\cos n\varphi d\varphi\right]\cdot \tag{III.8}$$

Both integrals can be expressed in terms of Bessel functions, for the operations of integration and differentiation with respect to z, applied to the equation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(z\sin\varphi/2)\cos n\varphi d\varphi=J_{2n}(z), \tag{III.9}$$

yield

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(z\sin\varphi/2)}{\sin\varphi/2}\cos n\varphi d\varphi=\int_0^z J_{2n}(x)dx, \tag{III.10}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(z\sin\varphi/2)\times\sin\varphi/2\cos n\varphi d\varphi=-J_{2n}'(z). \tag{III.11}$$

Therefore

$$P_n=n\omega_0\frac{e^2}{R}\left[2\beta^2J_{2n}'(2n\beta)-(1-\beta^2)\int_0^{2n\beta}J_{2n}(x)dx\right]\cdot \tag{III.12}$$

If the electron velocity is small compared to that of light, the dominant term in P_n is

$$\beta\ll 1: P_n=2\omega_0\frac{e^2}{R}(n+1)\frac{n^{2n+1}}{(2n+1)!}\beta^{2n+1}, \tag{III.13}$$

which makes it evident that appreciable power is radiated only into the fundamental frequency. For very large electron energies, $1-\beta^2\ll 1$, and we may place β equal to unity provided n is small compared with a critical harmonic number $n_c\sim(1-\beta^2)^{-\frac{3}{2}}$. Under these circumstances, we have

$$P_n=2n\omega_0\frac{e^2}{R}J_{2n}'(2n). \tag{III.14}$$

[9] See G. A. Schott, *Electromagnetic Radiation* (Cambridge University Press, Cambridge, 1912), pp. 109, 110.

The derivative of the Bessel function with equal order and argument can be approximated by its asymptotic form:[10]

$$J_{2n}'(2n) \sim \frac{3^{1/6}}{2\pi}\Gamma(\tfrac{2}{3})n^{-\frac{2}{3}}, \qquad \text{(III.15)}$$

with an error that is only 15 percent for $n=1$, and 5 percent for $n=5$. Hence

$$1-\beta^2 \ll 1, \quad n \ll n_c: P_n \simeq \frac{3^{1/6}}{\pi}\Gamma(\tfrac{2}{3})\omega_0 \frac{e^2}{R} n^{\frac{1}{3}}. \qquad \text{(III.16)}$$

Evidently the radiated power increases with increasing frequency. To find the behavior of P_n for very high frequencies, we must replace the Bessel functions in (III.12) by their asymptotic forms for large and comparable order and argument.[11] One thus obtains

$$n \gg 1: \int_0^{2n\beta} J_{2n}(x)dx = \frac{1}{3^{\frac{1}{2}}\pi}\int_{n/n_c}^{\infty} K_{1/3}(\eta)d\eta,$$
$$J_{2n}'(2n\beta) = \frac{1}{3^{\frac{1}{2}}\pi}\left(\frac{3}{2n_c}\right)^{\frac{2}{3}} K_{2/3}\left(\frac{n}{n_c}\right), \qquad \text{(III.17)}$$

with

$$n_c = \tfrac{3}{2}(1-\beta^2)^{-\frac{3}{2}}, \qquad \text{(III.18)}$$

whence

$$P_n = \frac{3^{\frac{1}{2}}}{4\pi}\omega_0 \frac{e^2}{R}\frac{1}{1-\beta^2}\frac{n}{n_c^2}\int_{n/n_c}^{\infty} K_{5/3}(\eta)d\eta, \qquad \text{(III.19)}$$

on employing the relation (II.15). The discrete spectrum is effectively a continuum if $n \gg 1$. The power radiated into a unit angular frequency interval is

$$P(\omega) = P_{\omega/\omega_0}\frac{1}{\omega_0} \qquad \text{(III.20)}$$

which yields an expression for $P(\omega)$ that is identical with (II.16), the more general result obtained for a high energy electron traversing an arbitrary trajectory.

The combined spectral and angular distribution function, $P(\mathbf{n}, \omega)$, also assumes the form characteristic of a discrete harmonic spectrum:

$$P(\mathbf{n}, \omega) = \sum_{n=1}^{\infty} \delta(\omega - n\omega_0)P_n(\mathbf{n}) \qquad \text{(III.21)}$$

on introducing the periodic nature of the electronic motion. Here

$$P_n(\mathbf{n}) = -\frac{e^2}{4\pi^2}\frac{n^2\omega_0^3}{c}\oint\left(1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)\right)$$
$$\times \cos n\omega_0\left(\tau - \frac{1}{c}\mathbf{n}\cdot(\mathbf{R}(t+\tau)-\mathbf{R}(t))\right)d\tau, \qquad \text{(III.22)}$$

in which the integration is extended over one traversal of the orbit. This quantity still involves the arbitrary initial phase of the electron in its circular path. An average with respect to t is equivalent to averaging over the initial phase. Further, the integrations with respect to t and $t+\tau=t'$ can be performed independently, which brings (III.22) into the form

$$P_n(\mathbf{n}) = -\frac{e^2}{8\pi^3}\frac{n^2\omega_0^4}{c}\oint\left(1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t')\right)$$
$$\times \exp\left[in\omega_0\left(t'-t-\frac{1}{c}\mathbf{n}\cdot(R(t')-\mathbf{R}(t))\right)\right]dtdt', \qquad \text{(III.23)}$$

in which it is understood that the real part is to be taken. We may now conveniently introduce a special coordinate system in which the xy plane is that of the orbit, and the unit vector $\mathbf{n}$ lies in the xz plane, at an angle ψ relative to the orbital plane. In terms of the polar coordinates φ and φ' for the points $\mathbf{R}(t)$ and $\mathbf{R}(t')$, we have

$$\mathbf{n}\cdot\mathbf{R}(t) = R\cos\psi\cos\varphi, \quad \mathbf{n}\cdot\mathbf{R}(t') = R\cos\psi\cos\varphi'$$
$$\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t') = \beta^2\cos(\varphi-\varphi'),$$

and

$$\omega_0 dt = d\varphi, \quad \omega_0 dt' = d\varphi',$$

whence

$$P_n(\psi) = \frac{e^2}{8\pi^3}\frac{n^2\omega_0^2}{c}$$
$$\times \int_0^{2\pi}(\beta^2\cos\varphi\cos\varphi'$$
$$+\beta^2\sin\varphi\sin\varphi' - 1)$$
$$\times \exp[in\beta\cos\psi\cos\varphi - in\varphi]$$
$$\times \exp[-in\beta\cos\psi\cos\varphi' + in\varphi']d\varphi d\varphi'. \qquad \text{(III.24)}$$

The integrals can be expressed in terms of Bessel

[10] Reference 7, p. 260.
[11] Reference 7, p. 248.

functions. From the well-known integral

$$\frac{1}{2\pi}\int_0^{2\pi} \exp[iz\cos\varphi - in\varphi]d\varphi = i^n J_n(z), \qquad \text{(III.25)}$$

we derive

$$\frac{1}{2\pi}\int_0^{2\pi} \exp[iz\cos\varphi - in\varphi] \times \cos\varphi d\varphi = i^{n-1}J_n'(z), \qquad \text{(III.26)}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \exp[iz\cos\varphi - in\varphi] \times \sin\varphi d\varphi = -i^n(n/z)J_n(z) \qquad \text{(III.27)}$$

by differentiation with respect to z and integration by parts, respectively. Therefore,

$$P_n(\psi) = \omega_0\frac{e^2}{R}\beta^3\frac{n^2}{2\pi}\left[(J_n'(n\beta\cos\psi))^2 + \sin^2\psi\left(\frac{J_n(n\beta\cos\psi)}{\beta\cos\psi}\right)^2\right], \qquad \text{(III.28)}$$

which yields the power radiated, in the nth harmonic, into a unit solid angle at the angle ψ with the orbital plane.

For $\beta\cos\psi \ll 1$, the leading term in (III.28) is

$$\frac{1}{2\pi}\omega_0\frac{e^2}{R}\beta\frac{(n\beta/2)^{2n}}{[(n-1)!]^2}(\cos\psi)^{2n-2}(1+\sin^2\psi),$$

which shows that the power radiated decreases rapidly with increasing harmonic number in either of two situations; emission by a low energy electron in any direction, or emission by a high energy electron in a direction almost normal to the orbital plane. On the other hand, the power radiated in the orbital plane by a very high energy electron is described by

$$1-\beta^2 \ll 1: P_n(0) = \omega_0\frac{e^2}{R}\frac{n^2}{2\pi}(J_n'(n))^2 \simeq \frac{6^{\frac{1}{3}}}{4\pi^3}(\Gamma(\tfrac{2}{3}))^2\omega_0\frac{e^2}{R}n^{\frac{2}{3}}, \qquad \text{(III.29)}$$

provided $n \ll n_c$. Evidently the power radiated in this direction increases with increasing frequency. To find the behavior at very high frequencies, without restriction to $\psi = 0$, we must employ the asymptotic forms of the Bessel functions for large order and argument:

$$1-\beta^2 \ll 1, \quad \psi^2 \ll 1, \quad n \gg 1:$$

$$J_n(n\beta\cos\psi) = \frac{1}{3^{\frac{1}{2}}\pi}(1-\beta^2+\psi^2)^{\frac{1}{2}}K_{1/3}(\xi),$$

$$J_n'(n\beta\cos\psi) = \frac{1}{3^{\frac{1}{2}}\pi}(1-\beta^2+\psi^2)\,K_{2/3}(\xi), \qquad \text{(III.30)}$$

where

$$\xi = \frac{n}{n_c}\left(1+\frac{\psi^2}{1-\beta^2}\right)^{\frac{3}{2}}. \qquad \text{(III.31)}$$

We obtain

$$P_n(\psi) = \frac{1}{6\pi^2}\omega_0\frac{e^2}{R}n^2(1-\beta^2+\psi^2)^2 \times\left[K_{2/3}^2(\xi) + \frac{\psi^2}{1-\beta^2+\psi^2}K_{1/3}^2(\xi)\right]. \qquad \text{(III.32)}$$

On replacing this quantity by

$$P(\psi, \omega) = (2\pi/\omega_0)P_{\omega/\omega_0}(\psi), \qquad \text{(III.33)}$$

the power radiated into a unit angular frequency range and a unit angular interval (rather than unit solid angle), we encounter a formula that is identical with (II.32), or (II.34), the more general result obtained for a high energy electron traversing an arbitrary trajectory.